Paper 1, Section I
3D Complex Analysis or Complex Methods

Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, be an analytic function of z in a domain D of the complex plane. Derive the Cauchy–Riemann equations relating the partial derivatives of u and v.

For $u = e^{-x}\cos y$, find v and hence $f(z)$.

Paper 1, Section II
13D Complex Analysis or Complex Methods

Consider the real function $f(t)$ of a real variable t defined by the following contour integral in the complex s-plane:

$$f(t) = \frac{1}{2\pi i}\int_\Gamma \frac{e^{st}}{(s^2+1)s^{1/2}}\,ds,$$

where the contour Γ is the line $s = \gamma + iy$, $-\infty < y < \infty$, for constant $\gamma > 0$. By closing the contour appropriately, show that

$$f(t) = \sin(t - \pi/4) + \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}dr}{(r^2+1)r^{1/2}}$$

when $t > 0$ and is zero when $t < 0$. You should justify your evaluation of the inversion integral over all parts of the contour.

By expanding $(r^2+1)^{-1}\ r^{-1/2}$ as a power series in r, and assuming that you may integrate the series term by term, show that the two leading terms, as $t \to \infty$, are

$$f(t) \sim \sin(t - \pi/4) + \frac{1}{(\pi t)^{1/2}} + \cdots .$$

[*You may assume that* $\int_0^\infty x^{-1/2}e^{-x}dx = \pi^{1/2}$.]

Paper 2, Section II

14D Complex Analysis or Complex Methods

Show that both the following transformations from the z-plane to the ζ-plane are conformal, except at certain critical points which should be identified in both planes, and in each case find a domain in the z-plane that is mapped onto the upper half ζ-plane:

$$\begin{aligned} \text{(i) } \zeta &= z + \frac{b^2}{z}\,; \\ \text{(ii) } \zeta &= \cosh \frac{\pi z}{b}\,, \end{aligned}$$

where b is real and positive.

Paper 3, Section I
5D Complex Methods

Use the residue calculus to evaluate

$$\text{(i)} \quad \oint_C z e^{1/z} dz \quad \text{and} \quad \text{(ii)} \quad \oint_C \frac{z dz}{1-4z^2}\,,$$

where C is the circle $|z|=1$.

Paper 4, Section II
15D Complex Methods

The function $u(x,y)$ satisfies Laplace's equation in the half-space $y \geqslant 0$, together with boundary conditions

$$u(x,y) \to 0 \text{ as } y \to \infty \text{ for all } x\,,$$
$$u(x,0) = u_0(x), \text{ where } x\,u_0(x) \to 0 \text{ as } |x| \to \infty.$$

Using Fourier transforms, show that

$$u(x,y) = \int_{-\infty}^{\infty} u_0(t) v(x-t,y) dt\,,$$

where

$$v(x,y) = \frac{y}{\pi(x^2+y^2)}\,.$$

Suppose that $u_0(x) = (x^2+a^2)^{-1}$. Using contour integration and the convolution theorem, or otherwise, show that

$$u(x,y) = \frac{y+a}{a\,[x^2+(y+a)^2]}\,.$$

[*You may assume the convolution theorem of Fourier transforms, i.e. that if* $\tilde{f}(k), \tilde{g}(k)$ *are the Fourier transforms of two functions* $f(x), g(x)$*, then* $\tilde{f}(k)\tilde{g}(k)$ *is the Fourier transform of* $\int_{-\infty}^{\infty} f(t)g(x-t)dt$.]