1/I/3C **Complex Analysis or Complex Methods**

Given that $f(z)$ is an analytic function, show that the mapping $w = f(z)$

(a) preserves angles between smooth curves intersecting at z if $f'(z) \neq 0$;

(b) has Jacobian given by $|f'(z)|^2$.

1/II/13C **Complex Analysis or Complex Methods**

By a suitable choice of contour show the following:

(a)

$$\int_0^\infty \frac{x^{1/n}}{1+x^2}\,dx = \frac{\pi}{2\cos(\pi/2n)}\,,$$

where $n > 1$,

(b)

$$\int_0^\infty \frac{x^{1/2}\log x}{1+x^2}\,dx = \frac{\pi^2}{2\sqrt{2}}\,.$$

2/II/14C **Complex Analysis or Complex Methods**

Let $f(z) = 1/(e^z - 1)$. Find the first three terms in the Laurent expansion for $f(z)$ valid for $0 < |z| < 2\pi$.

Now let n be a positive integer, and define

$$f_1(z) = \frac{1}{z} + \sum_{r=1}^{n} \frac{2z}{z^2 + 4\pi^2 r^2}\,,$$

$$f_2(z) = f(z) - f_1(z)\,.$$

Show that the singularities of f_2 in $\{z : |z| < 2(n+1)\pi\}$ are all removable. By expanding f_1 as a Laurent series valid for $|z| > 2n\pi$, and f_2 as a Taylor series valid for $|z| < 2(n+1)\pi$, find the coefficients of z^j for $-1 \leq j \leq 1$ in the Laurent series for f valid for $2n\pi < |z| < 2(n+1)\pi$.

By estimating an appropriate integral around the contour $|z| = (2n+1)\pi$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}\,.$$

3/II/14E **Complex Analysis**

State and prove Rouché's theorem, and use it to count the number of zeros of $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ inside the annulus $\{z : 1 < |z| < 2\}$.

Let $(p_n)_{n=1}^{\infty}$ be a sequence of polynomials of degree at most d with the property that $p_n(z)$ converges uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Prove that there is a polynomial p of degree at most d such that $p_n \to p$ uniformly on compact subsets of $\mathbb{C}$. [*If you use any results about uniform convergence of analytic functions, you should prove them.*]

Suppose that p has d distinct roots $z_1, \ldots, z_d$. Using Rouché's theorem, or otherwise, show that for each i there is a sequence $(z_{i,n})_{n=1}^{\infty}$ such that $p_n(z_{i,n}) = 0$ and $z_{i,n} \to z_i$ as $n \to \infty$.

4/I/4E **Complex Analysis**

Suppose that f and g are two functions which are analytic on the whole complex plane $\mathbb{C}$. Suppose that there is a sequence of distinct points $z_1, z_2, \ldots$ with $|z_i| \leqslant 1$ such that $f(z_i) = g(z_i)$. Show that $f(z) = g(z)$ for all $z \in \mathbb{C}$. [*You may assume any results on Taylor expansions you need, provided they are clearly stated.*]

What happens if the assumption that $|z_i| \leqslant 1$ is dropped?

3/I/5C Complex Methods

Using the contour integration formula for the inversion of Laplace transforms find the inverse Laplace transforms of the following functions:

$$\text{(a)} \quad \frac{s}{s^2+a^2} \quad (a \text{ real and non–zero}), \qquad \text{(b)} \quad \frac{1}{\sqrt{s}}\,.$$

[*You may use the fact that* $\int_{-\infty}^{\infty} e^{-bx^2}\,dx \;=\; \sqrt{\pi/b}$.]

4/II/15C Complex Methods

Let H be the domain $\mathbb{C} - \{x+iy : x \leq 0, y = 0\}$ (i.e., $\mathbb{C}$ cut along the negative x-axis). Show, by a suitable choice of branch, that the mapping

$$z \mapsto w \;=\; -i \log z$$

maps H onto the strip $S = \{z = x+iy, -\pi < x < \pi\}$.

How would a different choice of branch change the result?

Let G be the domain $\{z \in \mathbb{C} : |z| < 1, |z+i| > \sqrt{2}\}$. Find an analytic transformation that maps G to S, where S is the strip defined above.