COMPLEX METHODS 2

1 Evaluate $\oint_\gamma \bar{z}\, dz$ and $\oint_\gamma z^{\frac{1}{2}}\, dz$ (use the principal branch of $z^{\frac{1}{2}}$) in the two cases (i) γ is the circle $|z| = 1$ and (ii) γ is the circle $|z - 1| = 1$.

2 Find the first two non–vanishing coefficients in the Taylor expansion about the origin of the following functions, assuming principal branches for (i), (ii) and (iii). Where appropriate, you may make use of standard series expansions for $\log(1 + z)$, etc.

$$\text{(i)}\ \ z/\log(1+z)\ ; \qquad \text{(ii)}\ \ \sqrt{\cos z} - 1\ ; \qquad \text{(iii)}\ \ \log(1+e^z)\ ; \qquad \text{(iv)}\ \ e^{e^z}\ .$$

How would your answer differ, in each case, if you assumed branches different from the principal branch? State the range of values of z for which each series converges.

3 Use partial fractions to find the Laurent expansions of $1/((z - a)(z - b))$ about $z = 0$, where $|b| > |a| > 0$, in each of the regions $|z| < |a|$, $|a| < |z| < |b|$ and $|z| > |b|$.

4 Write down the positions in the complex plane and the types of the singularities of the following functions:

$$\text{(i)}\ \ \frac{1}{z^3(z-1)^2} \qquad \text{(ii)}\ \ \tan z \qquad \text{(iii)}\ \ z\coth z$$

$$\text{(iv)}\ \ \frac{e^z - e}{(1-z)^3} \qquad \text{(v)}\ \ \exp(\tan z) \qquad \text{(vi)}\ \ \log(1+e^z) \qquad \text{(vii)}\ \ \tan(z^{-1}).$$

How does $\tan z$ behave as $|z| \to \infty$ with $\mathrm{Im}\ z \neq 0$?

5 Find the first three terms of the Laurent expansion of $1/(\cos z - 1)$ valid for $|z| < 2\pi$. Find also the three non-zero central terms of the Laurent expansion valid for $2\pi < |z| < 4\pi$.

6 Evaluate, using Cauchy's theorem or the residue theorem:

(i) $\displaystyle\oint_C \frac{dz}{1+z^2}$ where C is the ellipse $x^2 + 4y^2 = 1$;

(ii) $\displaystyle\oint_C \frac{dz}{1+z^2}$ where C is the circle $x^2 + y^2 = 2$;

(iii) $\displaystyle\oint_C \frac{e^z \cos z\ dz}{(1+z^2)\sin z}$ where C is the circle $|z - (2+i)| = \sqrt{2}$.

7 Evaluate

$$\oint_C \frac{z^3\, e^{1/z}\, dz}{1+z}\ ,$$

where C is the circle $|z| = 2$.

8 (i) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ by closing the contour in the upper half-plane. How does the calculation differ if you close in the lower half-plane?

(ii) Evaluate $\lim_{R\to\infty} \int_{-R}^{R} \frac{x\,dx}{1+x+x^2}$. Why is the limit necessary?

(iii) Evaluate $\int_{-\infty}^{\infty} \frac{e^{ikx}dx}{1+x^2}$ for $k>0$ and for $k<0$.

(iv) Attempt to evaluate $\int_{-\infty}^{\infty} \frac{xe^{-x^2}dx}{1+x^2}$ by means of the residue theorem. Why did your method(s) fail? What is the value of the integral?

9 By integrating round a key-hole contour, show that

$$\int_0^\infty \frac{x^{a-1}dx}{1+x} = \frac{\pi}{\sin(\pi a)} \qquad (0<a<1).$$

Explain why the given restriction on the value of a is necessary.

10 By integrating round a contour involving the real axis and the line $z = s\exp(2\pi i/n)$ $(0 \le s < \infty)$, evaluate

$$\int_0^\infty \frac{dx}{1+x^n} \qquad (n \ge 2).$$

Check (by change of variable) that your answer agrees with that of the previous question.

11 Define

$$I = \int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} \quad . \qquad (\,b>a>0)$$

By evaluating $\oint_C z^{-1}dz$, where C is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ ,$$

show that $I = 2\pi/ab$.

12 Establish the following:

(i) $\displaystyle\int_0^\infty \frac{\cos x\ dx}{(1+x^2)^3} = \frac{7\pi}{16e}$; (ii) $\displaystyle\int_0^\infty \frac{\sin^2 x\ dx}{x^2} = \frac{\pi}{2}$; (iii) $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\ dx = 0$.

[*For part (iii), integrate* $(1+z^2)^{-1}(\log z)^2$ *round a keyhole, or* $(1+z^2)^{-1}\log z$ *along the real axis (or both).*]

13 Let $P(z)$ be a non-constant polynomial. Consider the contour integral

$$I = \oint_C \frac{P'(z)}{P(z)}\, dz\ .$$

Show that, if C is a contour that encloses no zeros of P, then $I = 0$. Evaluate $\lim_{R\to\infty} I$, where C is the circle $|z| = R$, and deduce that P has at least one zero in the complex plane.

14 By considering the integral of $f(z) = \dfrac{\cot z}{z^2 + \pi^2 a^2}$ around a suitable large contour, prove that (provided ia is not an integer)

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth\,(\pi a).$$

By considering a similar integral, prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2(\pi a)}\quad .$$

Justify the limiting operation needed as $a \to 0$ in order to deduce from each of these the value of $\sum_{n=1}^{\infty} 1/n^2$.

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