COMPLEX METHODS 1

1 (i) Let $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations. Define the function $g(z, \bar{z})$ by

$$g(z, \bar{z}) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Use the chain rule to show that $\partial g/\partial \bar{z} = 0$. Explain the significance of this result for analytic functions.

(ii) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$\text{Im z}\,; \qquad |z|^2\,; \qquad \operatorname{sech} z.$$

(iii) Let $f(z) = z^5/|z|^4$ $(z \neq 0)$ and $f(0) = 0$. Show that the real and imaginary parts of f satisfy the Cauchy-Riemann equations at $z = 0$, but that f is not differentiable at $z = 0$.
[When calculating, e.g., $u_x(0, 0)$, set $y = 0$ *before* differentiating with respect to x.]

2 Find *by inspection* complex analytic functions whose real parts are the following:

(i) x	(ii) xy	(iii) $\sin x \cosh y$
(iv) $\log(x^2 + y^2)$	(v) $\dfrac{y}{(x+1)^2 + y^2}$	(vi) $\tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$.

Deduce that the above functions are harmonic and give the domain on which they are harmonic.

3 Verify that the function $\phi(x, y) = e^x(x \cos y - y \sin y)$ is harmonic. Find its harmonic conjugate and determine the family of curves orthogonal to $\phi(x, y) = \alpha$, for an arbitrary constant α. Find the analytic function $f(z)$ such that Re $f = \phi$. Can the expression $f(z) = \phi(z, 0)$ be used to determine $f(z)$ in general?

4 Show that if $\phi(x, y)$ is a (planar) two-dimensional electrostatic potential, the corresponding electric field $\mathbf{E} = -\nabla\phi$ has magnitude $|f'(z)|$, where $f(z)$ is the analytic function such that $\phi = \text{Re} f$. Show that $\mathbf{E}$ makes an angle $\pi - arg f'(z)$ with the x-axis.

The potential for a line of charge of strength q per unit length passing through the origin, perpendicular to the z-plane is $\phi = -\frac{q}{2\pi\epsilon_0} \log r$, where r is the distance from the origin. Find the harmonic conjugate of ϕ and the electric field. Sketch the equipotential curves as well as the field lines.

5 The isothermal lines of a steady state temperature field are the family of curves $x^2 + y^2 = \alpha$, $\alpha > 0$. Find the general expression for the temperature, its harmonic conjugate and the equation for the family of flux lines. Sketch the isothermal and heat flow lines.

6 Show how the principal branch of $\log z$ can be used to define a branch of z^i which is single-valued on the set $\mathcal{D}$ defined by $\mathcal{D} = \mathcal{C} \setminus \{x+iy \ : \ x \leq 0\,,\ y = 0\}$ (i.e., the complex plane minus the negative real axis). Evaluate i^i for this branch. Show, using polar coordinates, that the branch of z^i defined above maps $\mathcal{D}$ onto an annulus which is covered infinitely often.

7 Show that $f(z) = [(z+1)\ (z-2)]^{1/3}$ has three branch points. Draw the possible branch cuts in the complex plane and the Riemann sphere.

8 Let $f(z)$ be the branch of $\log\left(\frac{1+z}{1-z}\right)$ defined by branch cuts on the real axis from $-\infty$ to -1 and from 1 to ∞, with $f(0) = 0$. Show, by considering the circle $z = e^{i\theta}$, that f maps the disc $\{|z| < 1\}$ onto the infinite strip $\{x + iy \ : \ -\pi/2 < y < \pi/2\}$.

Derive this result by breaking the mapping into a sequence of mappings $f_1(z) = 1 - z$, $f_2(z) = 2/z$, $f_3(z) = z - 1$, $f_4(z) = \log z$.

9 How does the disc $|z - 1| < 1$ transform under the mapping $z \to z^{-1}$?
Use the identity

$$\frac{z}{(z-1)^2} = \left(\frac{1}{1-z} - \frac{1}{2}\right)^2 - \frac{1}{4}$$

to show that the map $f(z) = z/(z-1)^2$ is a one-to-one conformal mapping of the disc $|z| < 1$ onto the domain $\mathcal{C} \setminus \{x + iy \ : \ x \leq -1/4\,,\ y = 0\}$.

10 Let

$$g(z) = \exp(\pi z/a), \quad h(z) = \sin(\pi z/a)\ .$$

Show that g maps

$$\{x + iy \ : \ 0 < y < a\} \text{ onto } \{x + iy \ : \ y > 0\}$$

and h maps

$$\{x + iy \ : \ -a/2 < x < a/2,\ y > 0\} \text{ onto } \{x + iy \ : \ y > 0\}.$$

Find a conformal map of $\{x + iy \ : \ -a/2 < x < a/2,\ y > 0\}$ onto $\{x + iy \ : \ 0 < y < a\}$. Find a function v which is harmonic on the strip $-a/2 < x < a/2,\ y > 0$ with limiting values on the boundaries given by: $v = 0$ on parts of the boundary in the left half plane $(x < 0)$ and $v = 1$ on parts of the boundary in the right half plane. Is there only one such function?

11 Show that $Arc\tan\frac{2x}{x^2+y^2-1}$ is harmonic. *Hint.* Consider $w(z) = \frac{i+z}{i-z}$ and $h(w) = \log w$.

12 *Joukowski map.* Consider the map $f(z) = \frac{1}{2}\ (z + 1/z)$. Find the points where the map is not conformal and determine how the angles between two vectors at those points change at their image. Show that this map takes concentric circles with radius $r > 1$ centered at the origin to cofocal ellipses. What is the image of the unit circle? Find the image of a circle with center on the x-axis, passing through the point $z = 1$ with $z = -1$ as an interior point. Discuss any possible application to aerodynamics.

For comments and corrections: gwg1@damtp.cam.ac.uk.