Paper 1, Section I

3F Analysis I

(a) State, without proof, the Bolzano–Weierstrass Theorem.

(b) Give an example of a sequence that does not have a convergent subsequence.

(c) Give an example of an unbounded sequence having a convergent subsequence.

(d) Let $a_n = 1 + (-1)^{\lfloor n/2 \rfloor}\big(1 + 1/n\big)$, where $\lfloor x \rfloor$ denotes the integer part of x. Find all values c such that the sequence $\{a_n\}$ has a subsequence converging to c. For each such value, provide a subsequence converging to it.

Paper 1, Section I

4D Analysis I

Find the radius of convergence of each of the following power series.

(i) $\displaystyle\sum_{n \geqslant 1} n^2 z^n$

(ii) $\displaystyle\sum_{n \geqslant 1} n^{n^{1/3}} z^n$

Paper 1, Section II

9F Analysis I

(a) State, without proof, the ratio test for the series $\displaystyle\sum_{n \geqslant 1} a_n$, where $a_n > 0$. Give examples, without proof, to show that, when $a_{n+1} < a_n$ and $a_{n+1}\big/a_n \to 1$, the series may converge or diverge.

(b) Prove that $\displaystyle\sum_{k=1}^{n-1} \frac{1}{k} \geqslant \log\ n$.

(c) Now suppose that $a_n > 0$ and that, for n large enough, $\dfrac{a_{n+1}}{a_n} \leqslant 1 - \dfrac{c}{n}$ where $c > 1$. Prove that the series $\displaystyle\sum_{n \geqslant 1} a_n$ converges.

[*You may find it helpful to prove the inequality* $\log\,(1 - x) < -x$ *for* $0 < x < 1$.]

Paper 1, Section II

10E Analysis I

State and prove the Intermediate Value Theorem.

A *fixed point* of a function $f : X \to X$ is an $x \in X$ with $f(x) = x$. Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Answer the following questions with justification.

(i) Does every continuous function $f : (0,1) \to (0,1)$ have a fixed point?

(ii) Does every continuous function $f : \mathbb{R} \to \mathbb{R}$ have a fixed point?

(iii) Does every function $f : [0,1] \to [0,1]$ (not necessarily continuous) have a fixed point?

(iv) Let $f : [0,1] \to [0,1]$ be a continuous function with $f(0) = 1$ and $f(1) = 0$. Can f have exactly two fixed points?

Paper 1, Section II

11E Analysis I

For each of the following two functions $f : \mathbb{R} \to \mathbb{R}$, determine the set of points at which f is continuous, and also the set of points at which f is differentiable.

$$\text{(i)}\ \ f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q}\ , \end{cases}$$

$$\text{(ii)}\ \ f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\ . \end{cases}$$

By modifying the function in (i), or otherwise, find a function (not necessarily continuous) $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at 0 and nowhere else.

Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at the points $1/2, 1/3, 1/4, \ldots$, but is differentiable at all other points.

Paper 1, Section II

12D Analysis I

State and prove the Fundamental Theorem of Calculus.

Let $f : [0,1] \to \mathbb{R}$ be integrable, and set $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for $0 < x < 1$. Must F be differentiable?

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must the Riemann integral of g from 0 to 1 exist?