**Paper 1, Section I**
**3D Analysis I**

Let $\sum_{n \geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have radius of convergence $R$, with $R \in [0, \infty]$.

Suppose the power series has radius of convergence $R$, with $0 < R < \infty$. Show that the sequence $|a_n z^n|$ is unbounded if $|z| > R$.

Find the radius of convergence of $\sum_{n \geqslant 1} z^n / n^3$.

**Paper 1, Section I**
**4E Analysis I**

Find the limit of each of the following sequences; justify your answers.

(i)
$$\frac{1 + 2 + \ldots + n}{n^2} \, ;$$

(ii)
$$\sqrt[n]{n} \, ;$$

(iii)
$$(a^n + b^n)^{1/n} \qquad \text{with} \qquad 0 < a \leqslant b \, .$$

**Paper 1, Section II**
**9E Analysis I**

Determine whether the following series converge or diverge. Any tests that you use should be carefully stated.

(a)

$$\sum_{n\geqslant 1}\frac{n!}{n^n}\ ;$$

(b)

$$\sum_{n\geqslant 1}\frac{1}{n+(\log n)^2}\ ;$$

(c)

$$\sum_{n\geqslant 1}\frac{(-1)^n}{1+\sqrt{n}}\ ;$$

(d)

$$\sum_{n\geqslant 1}\frac{(-1)^n}{n\,(2+(-1)^n)}\ .$$

**Paper 1, Section II**
**10F Analysis I**

(a) State and prove Taylor's theorem with the remainder in Lagrange's form.

(b) Suppose that $e:\ \mathbb{R}\to\mathbb{R}$ is a differentiable function such that $e(0)=1$ and $e'(x)=e(x)$ for all $x\in\mathbb{R}$. Use the result of (a) to prove that

$$e(x)=\sum_{n\geqslant 0}\frac{x^n}{n!}\quad\text{for all}\quad x\in\mathbb{R}.$$

[*No property of the exponential function may be assumed.*]

**Paper 1, Section II**
**11D Analysis I**

Define what it means for a bounded function $f : [a,\infty) \to \mathbb{R}$ to be Riemann integrable.

Show that a monotonic function $f : [a,b] \to \mathbb{R}$ is Riemann integrable, where $-\infty < a < b < \infty$.

Prove that if $f : [1,\infty) \to \mathbb{R}$ is a decreasing function with $f(x) \to 0$ as $x \to \infty$, then $\sum_{n\geqslant 1} f(n)$ and $\int_1^\infty f(x)\,dx$ either both diverge or both converge.

Hence determine, for $\alpha \in \mathbb{R}$, when $\sum_{n\geqslant 1} n^\alpha$ converges.

**Paper 1, Section II**
**12F Analysis I**

(a) Let $n \geqslant 1$ and $f$ be a function $\mathbb{R} \to \mathbb{R}$. Define carefully what it means for $f$ to be $n$ times differentiable at a point $x_0 \in \mathbb{R}$.

Set $\operatorname{sign}(x) = \begin{cases} x/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$

Consider the function $f(x)$ on the real line, with $f(0) = 0$ and

$$f(x) = x^2 \operatorname{sign}(x) \left|\cos\frac{\pi}{x}\right|, \quad x \neq 0.$$

(b) Is $f(x)$ differentiable at $x = 0$?

(c) Show that $f(x)$ has points of non-differentiability in any neighbourhood of $x = 0$.

(d) Prove that, in any finite interval $I$, the derivative $f'(x)$, at the points $x \in I$ where it exists, is bounded: $|f'(x)| \leqslant C$ where $C$ depends on $I$.