Paper 1, Section I

3F Analysis I

Determine the limits as $n \to \infty$ of the following sequences:

(a) $a_n = n - \sqrt{n^2 - n}$;

(b) $b_n = \cos^2\left(\pi\sqrt{n^2 + n}\right)$.

Paper 1, Section I

4E Analysis I

Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers. Prove that there exists $R \in [0, \infty]$ such that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

Give an example of a power series $\sum_{n=0}^{\infty} a_n z^n$ that diverges if $z = \pm 1$ and converges if $z = \pm \mathrm{i}$.

Paper 1, Section II

9F Analysis I

For each of the following series, determine for which real numbers x it diverges, for which it converges, and for which it converges absolutely. Justify your answers briefly.

(a) $$\sum_{n \geqslant 1} \frac{3 + \left(\sin x\right)^n}{n} \left(\sin x\right)^n,$$

(b) $$\sum_{n \geqslant 1} \left|\sin x\right|^n \frac{(-1)^n}{\sqrt{n}},$$

(c) $$\sum_{n \geqslant 1} \underbrace{\sin\left(0.99\ \sin\left(0.99 \ldots \sin\left(0.99\ x\right)\ldots\right)\right)}_{n\ \text{times}}.$$

Paper 1, Section II

10D Analysis I

State and prove the intermediate value theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $P = (a, b)$ be a point of the plane $\mathbb{R}^2$. Show that the set of distances from points $(x, f(x))$ on the graph of f to the point P is an interval $[A, \infty)$ for some value $A \geqslant 0$.

Paper 1, Section II

11D Analysis I

State and prove Rolle's theorem.

Let f and g be two continuous, real-valued functions on a closed, bounded interval $[a, b]$ that are differentiable on the open interval (a, b). By considering the determinant

$$\phi(x) = \begin{vmatrix} 1 & 1 & 0 \\ f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \end{vmatrix} = g(x)\,(f(b) - f(a)) - f(x)\,(g(b) - g(a)) \ ,$$

or otherwise, show that there is a point $c \in (a, b)$ with

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)) \ .$$

Suppose that $f, g : (0, \infty) \to \mathbb{R}$ are differentiable functions with $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Prove carefully that if the limit $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)} = \ell$ exists and is finite, then the limit $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ also exists and equals ℓ.

Paper 1, Section II

12E Analysis I

(a) What does it mean for a function $f : [a, b] \to \mathbb{R}$ to be *Riemann integrable*?

(b) Let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Suppose that for every $\delta > 0$ there is a sequence

$$0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_n < b_n \leqslant 1$$

such that for each i the function f is Riemann integrable on the closed interval $[a_i, b_i]$, and such that $\sum_{i=1}^{n}(b_i - a_i) \geqslant 1 - \delta$. Prove that f is Riemann integrable on $[0, 1]$.

(c) Let $f : [0, 1] \to \mathbb{R}$ be defined as follows. We set $f(x) = 1$ if x has an infinite decimal expansion that consists of 2s and 7s only, and otherwise we set $f(x) = 0$. Prove that f is Riemann integrable and determine $\int_0^1 f(x)\,\mathrm{d}x$.