1/I/3F Analysis I

State the ratio test for the convergence of a series.

Find all real numbers x such that the series

$$\sum_{n=1}^{\infty} \frac{x^n - 1}{n}$$

converges.

1/I/4E Analysis I

Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable, and for $0 \leqslant x \leqslant 1$ set $F(x) = \int_0^x f(t)\,dt$.

Assuming that f is continuous, prove that for every $0 < x < 1$ the function F is differentiable at x, with $F'(x) = f(x)$.

If we do not assume that f is continuous, must it still be true that F is differentiable at every $0 < x < 1$? Justify your answer.

1/II/9F Analysis I

Investigate the convergence of the series

(i) $$\sum_{n=2}^{\infty} \frac{1}{n^p (\log n)^q}$$

(ii) $$\sum_{n=3}^{\infty} \frac{1}{n\,(\log\log n)^r}$$

for positive real values of p, q and r.

[*You may assume that for any positive real value of* α, $\log n < n^\alpha$ *for* n *sufficiently large. You may assume standard tests for convergence, provided that they are clearly stated.*]

1/II/10D **Analysis I**

(a) State and prove the intermediate value theorem.

(b) An *interval* is a subset I of $\mathbb{R}$ with the property that if x and y belong to I and $x < z < y$ then z also belongs to I. Prove that if I is an interval and f is a continuous function from I to $\mathbb{R}$ then $f(I)$ is an interval.

(c) For each of the following three pairs (I, J) of intervals, either exhibit a continuous function f from I to $\mathbb{R}$ such that $f(I) = J$ or explain briefly why no such continuous function exists:

(i) $I = [0,1]$, $J = [0,\infty)$;

(ii) $I = (0,1]$, $J = [0,\infty)$;

(iii) $I = (0,1]$, $J = (-\infty,\infty)$.

1/II/11D **Analysis I**

(a) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ and suppose that both f and g are differentiable at the real number x. Prove that the product fg is also differentiable at x.

(b) Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $g(x) = x^2 f(x)$ for every x. Prove that g is differentiable at x if and only if either $x = 0$ or f is differentiable at x.

(c) Now let f be any continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $g(x) = f(x)^2$ for every x. Prove that g is differentiable at x if and only if at least one of the following two possibilities occurs:

(i) f is differentiable at x;

(ii) $f(x) = 0$ and

$$\frac{f(x+h)}{|h|^{1/2}} \longrightarrow 0 \quad \text{as} \quad h \to 0\,.$$

1/II/12E **Analysis I**

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Prove that there exists an $R \in [0, \infty]$ such that the series converges for every z with $|z| < R$ and diverges for every z with $|z| > R$.

Find the value of R for each of the following power series:

(i) $$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n ;$$

(ii) $$\sum_{n=0}^{\infty} z^{n!} .$$

In each case, determine at which points on the circle $|z| = R$ the series converges.