

**1/I/3F Analysis**

Prove that, for positive real numbers  $a$  and  $b$ ,

$$2\sqrt{ab} \leq a + b.$$

For positive real numbers  $a_1, a_2, \dots$ , prove that the convergence of

$$\sum_{n=1}^{\infty} a_n$$

implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}.$$

**1/I/4D Analysis**

Let  $\sum_{n=0}^{\infty} a_n z^n$  be a complex power series. Show that there exists  $R \in [0, \infty]$  such that  $\sum_{n=0}^{\infty} a_n z^n$  converges whenever  $|z| < R$  and diverges whenever  $|z| > R$ .

Find the value of  $R$  for the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

**1/II/9F Analysis**

Let  $a_1 = \sqrt{2}$ , and consider the sequence of positive real numbers defined by

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}}, \quad n = 1, 2, 3, \dots$$

Show that  $a_n \leq 2$  for all  $n$ . Prove that the sequence  $a_1, a_2, \dots$  converges to a limit.

Suppose instead that  $a_1 = 4$ . Prove that again the sequence  $a_1, a_2, \dots$  converges to a limit.

Prove that the limits obtained in the two cases are equal.

**1/II/10E Analysis**

State and prove the Mean Value Theorem.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that, for every  $x \in \mathbb{R}$ ,  $f''(x)$  exists and is non-negative.

(i) Show that if  $x \leq y$  then  $f'(x) \leq f'(y)$ .

(ii) Let  $\lambda \in (0, 1)$  and  $a < b$ . Show that there exist  $x$  and  $y$  such that

$$f(\lambda a + (1 - \lambda)b) = f(a) + (1 - \lambda)(b - a)f'(x) = f(b) - \lambda(b - a)f'(y)$$

and that

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

**1/II/11E Analysis**

Let  $a < b$  be real numbers, and let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Show that  $f$  is bounded on  $[a, b]$ , and that there exist  $c, d \in [a, b]$  such that for all  $x \in [a, b]$ ,  $f(c) \leq f(x) \leq f(d)$ .

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow -\infty} g(x) = 0.$$

Show that  $g$  is bounded. Show also that, if  $a$  and  $c$  are real numbers with  $0 < c \leq g(a)$ , then there exists  $x \in \mathbb{R}$  with  $g(x) = c$ .

**1/II/12D Analysis**

Explain carefully what it means to say that a bounded function  $f : [0, 1] \rightarrow \mathbb{R}$  is *Riemann integrable*.

Prove that every continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  is Riemann integrable.

For each of the following functions from  $[0, 1]$  to  $\mathbb{R}$ , determine with proof whether or not it is Riemann integrable:

(i) the function  $f(x) = x \sin \frac{1}{x}$  for  $x \neq 0$ , with  $f(0) = 0$ ;

(ii) the function  $g(x) = \sin \frac{1}{x}$  for  $x \neq 0$ , with  $g(0) = 0$ .