1/I/3F **Analysis**

Let $a_n \in \mathbb{R}$ for $n \geqslant 1$. What does it mean to say that the infinite series $\sum_n a_n$ converges to some value A? Let $s_n = a_1 + \cdots + a_n$ for all $n \geqslant 1$. Show that if $\sum_n a_n$ converges to some value A, then the sequence whose n-th term is

$$(s_1 + \cdots + s_n)/n$$

converges to some value $\tilde{A}$ as $n \to \infty$. Is it always true that $A = \tilde{A}$? Give an example where $(s_1 + \cdots + s_n)/n$ converges but $\sum_n a_n$ does not.

1/I/4D **Analysis**

Let $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ be power series in the complex plane with radii of convergence R and S respectively. Show that if $R \neq S$ then $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ has radius of convergence $\min(R, S)$. [*Any results on absolute convergence that you use should be clearly stated.*]

1/II/9E **Analysis**

State and prove the Intermediate Value Theorem.

Suppose that the function f is differentiable everywhere in some open interval containing $[a, b]$, and that $f'(a) < k < f'(b)$. By considering the functions g and h defined by

$$g(x) = \frac{f(x) - f(a)}{x - a} \quad (a < x \leqslant b) \ , \quad g(a) = f'(a)$$

and

$$h(x) = \frac{f(b) - f(x)}{b - x} \quad (a \leqslant x < b) \ , \quad h(b) = f'(b),$$

or otherwise, show that there is a subinterval $[a', b'] \subseteq [a, b]$ such that

$$\frac{f(b') - f(a')}{b' - a'} = k.$$

Deduce that there exists $c \in (a, b)$ with $f'(c) = k$. [*You may assume the Mean Value Theorem.*]

1/II/10E **Analysis**

Prove that if the function f is infinitely differentiable on an interval (r, s) containing a, then for any $x \in (r, s)$ and any positive integer n we may expand $f(x)$ in the form

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(f, a, x),$$

where the remainder term $R_n(f, a, x)$ should be specified explicitly in terms of $f^{(n+1)}$.

Let $p(t)$ be a nonzero polynomial in t, and let f be the real function defined by

$$f(x) = p\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) \quad (x \neq 0)\ , \quad f(0) = 0.$$

Show that f is differentiable everywhere and that

$$f'(x) = q\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) \quad (x \neq 0)\ , \quad f'(0) = 0,$$

where $q(t) = 2t^3p(t) - t^2p'(t)$. Deduce that f is infinitely differentiable, but that there exist arbitrarily small values of x for which the remainder term $R_n(f, 0, x)$ in the Taylor expansion of f about 0 does not tend to 0 as $n \to \infty$.

1/II/11F **Analysis**

Consider a sequence $(a_n)_{n\geqslant 1}$ of real numbers. What does it mean to say that $a_n \to a \in \mathbb{R}$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$? What does it mean to say that $a_n \to -\infty$ as $n \to \infty$? Show that for every sequence of real numbers there exists a subsequence which converges to a value in $\mathbb{R} \cup \{\infty, -\infty\}$. [*You may use the Bolzano–Weierstrass theorem provided it is clearly stated.*]

Give an example of a bounded sequence $(a_n)_{n\geqslant 1}$ which is not convergent, but for which

$$a_{n+1} - a_n \to 0 \quad \text{as} \quad n \to \infty.$$

1/II/12D **Analysis**

Let f_1 and f_2 be Riemann integrable functions on $[a, b]$. Show that $f_1 + f_2$ is Riemann integrable.

Let f be a Riemann integrable function on $[a, b]$ and set $f^+(x) = \max(f(x), 0)$. Show that f^+ and $|f|$ are Riemann integrable.

Let f be a function on $[a, b]$ such that $|f|$ is Riemann integrable. Is it true that f is Riemann integrable? Justify your answer.

Show that if f_1 and f_2 are Riemann integrable on $[a, b]$, then so is $\max(f_1, f_2)$. Suppose now $f_1, f_2, \ldots$ is a sequence of Riemann integrable functions on $[a, b]$ and $f(x) = \sup_n f_n(x)$; is it true that f is Riemann integrable? Justify your answer.