1/I/3F Analysis

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

Let A denote a non-empty set of real numbers which has a supremum but no maximum. Show that for every $\epsilon > 0$ there are infinitely many elements of A contained in the open interval

$$(\sup A - \epsilon\,,\ \sup A).$$

Give an example of a non-empty set of real numbers which has a supremum *and* maximum and for which the above conclusion does not hold.

1/I/4D Analysis

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series in the complex plane with radius of convergence R. Show that $|a_n z^n|$ is unbounded in n for any z with $|z| > R$. State clearly any results on absolute convergence that are used.

For every $R \in [0, \infty]$, show that there exists a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R.

1/II/9F Analysis

Examine each of the following series and determine whether or not they converge. Give reasons in each case.

(i)
$$\sum_{n=1}^{\infty} \frac{1}{n^2}\,,$$

(ii)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + (-1)^{n+1}\, 2n + 1}\,,$$

(iii)
$$\sum_{n=1}^{\infty} \frac{n^3 + (-1)^n\, 8n^2 + 1}{n^4 + (-1)^{n+1}\, n^2}\,,$$

(iv)
$$\sum_{n=1}^{\infty} \frac{n^3}{e^{e^n}}\,.$$

1/II/10D Analysis

Explain what it means for a bounded function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function. Show that for each $x \in (0, \infty)$, there exists a unique point $g(x) \in (0, x)$ such that

$$\frac{1}{x} \int_0^x f(t)\, dt = f(g(x)).$$

Find $g(x)$ if $f(x) = e^{-x}$.

Suppose now that f is differentiable and $f'(x) < 0$ for all $x \in (0, \infty)$. Prove that g is differentiable at all $x \in (0, \infty)$ and $g'(x) > 0$ for all $x \in (0, \infty)$, stating clearly any results on the inverse of f you use.

1/II/11E Analysis

Prove that if f is a continuous function on the interval $[a, b]$ with $f(a) < 0 < f(b)$ then $f(c) = 0$ for some $c \in (a, b)$.

Let g be a continuous function on $[0, 1]$ satisfying $g(0) = g(1)$. By considering the function $f(x) = g(x + \frac{1}{2}) - g(x)$ on $[0, \frac{1}{2}]$, show that $g(c + \frac{1}{2}) = g(c)$ for some $c \in [0, \frac{1}{2}]$. Show, more generally, that for any positive integer n there exists a point $c_n \in [0, \frac{n-1}{n}]$ for which $g(c_n + \frac{1}{n}) = g(c_n)$.

1/II/12E Analysis

State and prove Rolle's Theorem.

Prove that if the real polynomial p of degree n has all its roots real (though not necessarily distinct), then so does its derivative p'. Give an example of a cubic polynomial p for which the converse fails.