1/I/1A **Algebra and Geometry**

(i) The spherical polar unit basis vectors $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$ in $\mathbb{R}^3$ are given in terms of the Cartesian unit basis vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ by

$$\begin{aligned}\mathbf{e}_r &= \mathbf{i}\cos\phi\sin\theta + \mathbf{j}\sin\phi\sin\theta + \mathbf{k}\cos\theta\,,\\ \mathbf{e}_\theta &= \mathbf{i}\cos\phi\cos\theta + \mathbf{j}\sin\phi\cos\theta - \mathbf{k}\sin\theta\,,\\ \mathbf{e}_\phi &= -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi\,.\end{aligned}$$

Express $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ in terms of $\mathbf{e}_r, \mathbf{e}_\phi$ and $\mathbf{e}_\theta$.

(ii) Use suffix notation to prove the following identity for the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $\mathbb{R}^3$:

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{A}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\,\mathbf{A}\,.$$

1/I/2B **Algebra and Geometry**

For the equations

$$\begin{aligned}px + y + z &= 1,\\ x + 2y + 4z &= t,\\ x + 4y + 10z &= t^2,\end{aligned}$$

find the values of p and t for which

(i) there is a unique solution;

(ii) there are infinitely many solutions;

(iii) there is no solution.

1/II/5B **Algebra and Geometry**

(i) Describe geometrically the following surfaces in three-dimensional space:

(a) $\mathbf{r}\cdot\mathbf{u} = \alpha|\mathbf{r}|$, where $0 < |\alpha| < 1$;

(b) $|\mathbf{r} - (\mathbf{r}\cdot\mathbf{u})\mathbf{u}| = \beta$, where $\beta > 0$.

Here α and β are fixed scalars and $\mathbf{u}$ is a fixed unit vector. You should identify the meaning of α, β and $\mathbf{u}$ for these surfaces.

(ii) The plane $\mathbf{n}\cdot\mathbf{r} = p$, where $\mathbf{n}$ is a fixed unit vector, and the sphere with centre $\mathbf{c}$ and radius a intersect in a circle with centre $\mathbf{b}$ and radius ρ.

(a) Show that $\mathbf{b} - \mathbf{c} = \lambda\mathbf{n}$, where you should give λ in terms of a and ρ.

(b) Find ρ in terms of $\mathbf{c}, \mathbf{n}, a$ and p.

1/II/6C Algebra and Geometry

Let $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\mathbf{x} \mapsto \mathbf{x}' = a\mathbf{x} + b(\mathbf{n} \times \mathbf{x})\,,$$

where a and b are positive scalar constants, and $\mathbf{n}$ is a unit vector.

(i) By considering the effect of $\mathcal{M}$ on $\mathbf{n}$ and on a vector orthogonal to $\mathbf{n}$, describe geometrically the action of $\mathcal{M}$.

(ii) Express the map $\mathcal{M}$ as a matrix M using suffix notation. Find a, b and $\mathbf{n}$ in the case

$$\mathrm{M} = \begin{pmatrix} 2 & -2 & 2 \\ 2 & 2 & -1 \\ -2 & 1 & 2 \end{pmatrix}.$$

(iii) Find, in the general case, the inverse map (i.e. express $\mathbf{x}$ in terms of $\mathbf{x}'$ in vector form).

1/II/7C Algebra and Geometry

Let $\mathbf{x}$ and $\mathbf{y}$ be non-zero vectors in a real vector space with scalar product denoted by $\mathbf{x} \cdot \mathbf{y}$. Prove that $(\mathbf{x} \cdot \mathbf{y})^{\mathbf{2}} \leq (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$, and prove also that $(\mathbf{x} \cdot \mathbf{y})^{\mathbf{2}} = (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y})$ if and only if $\mathbf{x} = \lambda \mathbf{y}$ for some scalar λ.

(i) By considering suitable vectors in $\mathbb{R}^3$, or otherwise, prove that the inequality $x^2 + y^2 + z^2 \geq yz + zx + xy$ holds for any real numbers x, y and z.

(ii) By considering suitable vectors in $\mathbb{R}^4$, or otherwise, show that only one choice of real numbers x, y, z satisfies $3(x^2 + y^2 + z^2 + 4) - 2(yz + zx + xy) - 4(x + y + z) = 0$, and find these numbers.

1/II/8A Algebra and Geometry

(i) Show that any line in the complex plane $\mathbb{C}$ can be represented in the form

$$\bar{c}z + c\bar{z} + r = 0\,,$$

where $c \in \mathbb{C}$ and $r \in \mathbb{R}$.

(ii) If z and u are two complex numbers for which

$$\left|\frac{z+u}{z+\bar{u}}\right| = 1\,,$$

show that either z or u is real.

(iii) Show that any Möbius transformation

$$w = \frac{az+b}{cz+d} \quad (bc - ad \neq 0)$$

that maps the real axis $z = \bar{z}$ into the unit circle $|w| = 1$ can be expressed in the form

$$w = \lambda \frac{z+k}{z+\bar{k}}\,,$$

where $\lambda, k \in \mathbb{C}$ and $|\lambda| = 1$.

3/I/1D Algebra and Geometry

Prove that every permutation of $\{1, \ldots, n\}$ may be expressed as a product of disjoint cycles.

Let $\sigma = (1234)$ and let $\tau = (345)(678)$. Write $\sigma\tau$ as a product of disjoint cycles. What is the order of $\sigma\tau$?

3/I/2D Algebra and Geometry

What does it mean to say that groups G and H are *isomorphic*?

Prove that no two of C_8, $C_4 \times C_2$ and $C_2 \times C_2 \times C_2$ are isomorphic. [Here C_n denotes the cyclic group of order n.]

Give, with justification, a group of order 8 that is not isomorphic to any of those three groups.

3/II/5D Algebra and Geometry

Let x be an element of a finite group G. What is meant by the *order* of x? Prove that the order of x must divide the order of G. [*No version of Lagrange's theorem or the Orbit-Stabilizer theorem may be used without proof.*]

If G is a group of order n, and d is a divisor of n with $d < n$, is it always true that G must contain an element of order d? Justify your answer.

Prove that if m and n are coprime then the group $C_m \times C_n$ is cyclic.

If m and n are not coprime, can it happen that $C_m \times C_n$ is cyclic?

[Here C_n denotes the cyclic group of order n.]

3/II/6D Algebra and Geometry

What does it mean to say that a subgroup H of a group G is *normal*? Give, with justification, an example of a subgroup of a group that is normal, and also an example of a subgroup of a group that is not normal.

If H is a normal subgroup of G, explain carefully how to make the set of (left) cosets of H into a group.

Let H be a normal subgroup of a finite group G. Which of the following are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) If G is cyclic then H and G/H are cyclic.

(ii) If H and G/H are cyclic then G is cyclic.

(iii) If G is abelian then H and G/H are abelian.

(iv) If H and G/H are abelian then G is abelian.

3/II/7D Algebra and Geometry

Let A be a real symmetric $n \times n$ matrix. Prove that every eigenvalue of A is real, and that eigenvectors corresponding to distinct eigenvalues are orthogonal. Indicate clearly where in your argument you have used the fact that A is real.

What does it mean to say that a real $n \times n$ matrix P is *orthogonal* ? Show that if P is orthogonal and A is as above then $P^{-1}AP$ is symmetric. If P is any real invertible matrix, must $P^{-1}AP$ be symmetric? Justify your answer.

Give, with justification, real 2×2 matrices B, C, D, E with the following properties:

(i) B has no real eigenvalues;

(ii) C is not diagonalisable over $\mathbb{C}$;

(iii) D is diagonalisable over $\mathbb{C}$, but not over $\mathbb{R}$;

(iv) E is diagonalisable over $\mathbb{R}$, but does not have an orthonormal basis of eigenvectors.

3/II/8D Algebra and Geometry

In the group of Möbius maps, what is the order of the Möbius map $z \mapsto \dfrac{1}{z}$? What is the order of the Möbius map $z \mapsto \dfrac{1}{1-z}$?

Prove that every Möbius map is conjugate either to a map of the form $z \mapsto \mu z$ (some $\mu \in \mathbb{C}$) or to the map $z \mapsto z+1$. Is $z \mapsto z+1$ conjugate to a map of the form $z \mapsto \mu z$?

Let f be a Möbius map of order n, for some positive integer n. Under the action on $\mathbb{C} \cup \{\infty\}$ of the group generated by f, what are the various sizes of the orbits? Justify your answer.