1/I/1B Algebra and Geometry

Consider the cone K in $\mathbb{R}^3$ defined by

$$x_3^2 = x_1^2 + x_2^2, \quad x_3 > 0.$$

Find a unit normal $\mathbf{n} = (n_1, n_2, n_3)$ to K at the point $\mathbf{x} = (x_1, x_2, x_3)$ such that $n_3 \geqslant 0$. Show that if $\mathbf{p} = (p_1, p_2, p_3)$ satisfies

$$p_3^2 \geqslant p_1^2 + p_2^2$$

and $p_3 \geqslant 0$ then

$$\mathbf{p} \cdot \mathbf{n} \geqslant 0.$$

1/I/2A Algebra and Geometry

Express the unit vector $\mathbf{e}_r$ of spherical polar coordinates in terms of the orthonormal Cartesian basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Express the equation for the paraboloid $z = x^2 + y^2$ in (i) cylindrical polar coordinates (ρ, ϕ, z) and (ii) spherical polar coordinates (r, θ, ϕ).

In spherical polar coordinates, a surface is defined by $r^2 \cos 2\theta = a$, where a is a real non-zero constant. Find the corresponding equation for this surface in Cartesian coordinates and sketch the surfaces in the two cases $a > 0$ and $a < 0$.

1/II/5C Algebra and Geometry

Prove the Cauchy–Schwarz inequality,

$$|\mathbf{x} \cdot \mathbf{y}| \leqslant |\mathbf{x}||\mathbf{y}|,$$

for two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. Under what condition does equality hold?

Consider a pyramid in $\mathbb{R}^n$ with vertices at the origin O and at $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_1 = (1, 0, 0, \ldots)$, $\mathbf{e}_2 = (0, 1, 0, \ldots)$, and so on. The "base" of the pyramid is the $(n-1)$-dimensional object B specified by $(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n) \cdot \mathbf{x} = 1$, $\mathbf{e}_i \cdot \mathbf{x} \geqslant 0$ for $i = 1, \ldots, n$.

Find the point C in B equidistant from each vertex of B and find the length of OC. (C is the centroid of B.)

Show, using the Cauchy–Schwarz inequality, that this is the closest point in B to the origin O.

Calculate the angle between OC and any edge of the pyramid connected to O. What happens to this angle and to the length of OC as n tends to infinity?

1/II/6C Algebra and Geometry

Given a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, write down the vector $\mathbf{x}'$ obtained by rotating $\mathbf{x}$ through an angle θ.

Given a unit vector $\mathbf{n} \in \mathbb{R}^3$, any vector $\mathbf{x} \in \mathbb{R}^3$ may be written as $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$ where $\mathbf{x}_\parallel$ is parallel to $\mathbf{n}$ and $\mathbf{x}_\perp$ is perpendicular to $\mathbf{n}$. Write down explicit formulae for $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$, in terms of $\mathbf{n}$ and $\mathbf{x}$. Hence, or otherwise, show that the linear map

$$\mathbf{x} \mapsto \mathbf{x}' = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \cos\theta(\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n}) + \sin\theta(\mathbf{n}\times\mathbf{x}) \qquad (*)$$

describes a rotation about $\mathbf{n}$ through an angle θ, in the positive sense defined by the right hand rule.

Write equation $(*)$ in matrix form, $x'_i = R_{ij}x_j$. Show that the trace $R_{ii} = 1+2\cos\theta$.

Given the rotation matrix

$$R = \frac{1}{2}\begin{pmatrix} 1+r & 1-r & 1 \\ 1-r & 1+r & -1 \\ -1 & 1 & 2r \end{pmatrix},$$

where $r = 1/\sqrt{2}$, find the two pairs $(\theta, \mathbf{n})$, with $-\pi \leqslant \theta < \pi$, giving rise to R. Explain why both represent the same rotation.

1/II/7B Algebra and Geometry

(i) Let $\mathbf{u}, \mathbf{v}$ be unit vectors in $\mathbb{R}^3$. Write the transformation on vectors $\mathbf{x} \in \mathbb{R}^3$

$$\mathbf{x} \mapsto (\mathbf{u}\cdot\mathbf{x})\mathbf{u} + \mathbf{v}\times\mathbf{x}$$

in matrix form as $\mathbf{x} \mapsto A\mathbf{x}$ for a matrix A. Find the eigenvalues in the two cases (a) when $\mathbf{u}\cdot\mathbf{v} = 0$, and (b) when $\mathbf{u}, \mathbf{v}$ are parallel.

(ii) Let $\mathcal{M}$ be the set of 2×2 complex hermitian matrices with trace zero. Show that if $A \in \mathcal{M}$ there is a unique vector $\mathbf{x} \in \mathbb{R}^3$ such that

$$A = \mathcal{R}(\mathbf{x}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Show that if U is a 2×2 unitary matrix, the transformation

$$A \mapsto U^{-1}AU$$

maps $\mathcal{M}$ to $\mathcal{M}$, and that if $U^{-1}\mathcal{R}(\mathbf{x})U = \mathcal{R}(\mathbf{y})$, then $\|\mathbf{x}\| = \|\mathbf{y}\|$ where $\|\cdot\|$ means ordinary Euclidean length. [*Hint: Consider determinants.*]

1/II/8A **Algebra and Geometry**

(i) State de Moivre's theorem. Use it to express $\cos 5\theta$ as a polynomial in $\cos\theta$.

(ii) Find the two fixed points of the Möbius transformation

$$z \longmapsto \omega = \frac{3z+1}{z+3}\,,$$

that is, find the two values of z for which $\omega = z$.

Given that $c \neq 0$ and $(a-d)^2 + 4bc \neq 0$, show that a general Möbius transformation

$$z \longmapsto \omega = \frac{az+b}{cz+d}\,, \qquad ad - bc \neq 0\,,$$

has two fixed points α, β given by

$$\alpha = \frac{a-d+m}{2c}\,, \qquad \beta = \frac{a-d-m}{2c}\,,$$

where $\pm m$ are the square roots of $(a-d)^2 + 4bc$.

Show that such a transformation can be expressed in the form

$$\frac{\omega-\alpha}{\omega-\beta} = k\frac{z-\alpha}{z-\beta}\,,$$

where k is a constant that you should determine.

3/I/1D **Algebra and Geometry**

Give an example of a real 3×3 matrix A with eigenvalues -1, $(1\pm i)/\sqrt{2}$. Prove or give a counterexample to the following statements:

(i) any such A is diagonalisable over $\mathbb{C}$;

(ii) any such A is orthogonal;

(iii) any such A is diagonalisable over $\mathbb{R}$.

3/I/2D **Algebra and Geometry**

Show that if H and K are subgroups of a group G, then $H \cap K$ is also a subgroup of G. Show also that if H and K have orders p and q respectively, where p and q are coprime, then $H \cap K$ contains only the identity element of G. [*You may use Lagrange's theorem provided it is clearly stated.*]

3/II/5D Algebra and Geometry

Let G be a group and let A be a non-empty subset of G. Show that

$$C(A) = \{g \in G : \; gh = hg \quad \text{for all} \;\; h \in A\}$$

is a subgroup of G.

Show that $\rho : G \times G \to G$ given by

$$\rho(g, h) = ghg^{-1}$$

defines an action of G on itself.

Suppose G is finite, let $O_1, \ldots, O_n$ be the orbits of the action ρ and let $h_i \in O_i$ for $i = 1, \ldots, n$. Using the Orbit–Stabilizer Theorem, or otherwise, show that

$$|G| = |C(G)| + \sum_i |G|/|C(\{h_i\})|$$

where the sum runs over all values of i such that $|O_i| > 1$.

Let G be a finite group of order p^r, where p is a prime and r is a positive integer. Show that $C(G)$ contains more than one element.

3/II/6D Algebra and Geometry

Let $\theta : G \to H$ be a homomorphism between two groups G and H. Show that the image of θ, $\theta(G)$, is a subgroup of H; show also that the kernel of θ, $\ker(\theta)$, is a normal subgroup of G.

Show that $G/\ker(\theta)$ is isomorphic to $\theta(G)$.

Let $O(3)$ be the group of 3×3 real orthogonal matrices and let $SO(3) \subset O(3)$ be the set of orthogonal matrices with determinant 1. Show that $SO(3)$ is a normal subgroup of $O(3)$ and that $O(3)/SO(3)$ is isomorphic to the cyclic group of order 2.

Give an example of a homomorphism from $O(3)$ to $SO(3)$ with kernel of order 2.

3/II/7D Algebra and Geometry

Let $SL(2,\mathbb{R})$ be the group of 2×2 real matrices with determinant 1 and let $\sigma : \mathbb{R} \to SL(2,\mathbb{R})$ be a homomorphism. On $K = \mathbb{R} \times \mathbb{R}^2$ consider the product

$$(x, \mathbf{v}) * (y, \mathbf{w}) = (x + y, \mathbf{v} + \sigma(x)\mathbf{w}).$$

Show that K with this product is a group.

Find the homomorphism or homomorphisms σ for which K is a commutative group.

Show that the homomorphisms σ for which the elements of the form $(0, \mathbf{v})$ with $\mathbf{v} = (a, 0)$, $a \in \mathbb{R}$, commute with every element of K are precisely those such that

$$\sigma(x) = \begin{pmatrix} 1 & r(x) \\ 0 & 1 \end{pmatrix},$$

with $r : (\mathbb{R}, +) \to (\mathbb{R}, +)$ an arbitrary homomorphism.

3/II/8D Algebra and Geometry

Show that every Möbius transformation can be expressed as a composition of maps of the forms: $S_1(z) = z + \alpha$, $S_2(z) = \lambda z$ and $S_3(z) = 1/z$, where $\alpha, \lambda \in \mathbb{C}$.

Show that if z_1, z_2, z_3 and w_1, w_2, w_3 are two triples of distinct points in $\mathbb{C} \cup \{\infty\}$, there exists a unique Möbius transformation that takes z_j to w_j $(j = 1, 2, 3)$.

Let G be the group of those Möbius transformations which map the set $\{0, 1, \infty\}$ to itself. Find all the elements of G. To which standard group is G isomorphic?