1/I/1C **Algebra and Geometry**

Convert the following expressions from suffix notation (assuming the summation convention in three dimensions) into standard notation using vectors and/or matrices, where possible, identifying the one expression that is incorrectly formed:

(i) δ_{ij},

(ii) $\delta_{ii}\,\delta_{ij}$,

(iii) $\delta_{ll}\,a_i\,b_j\,C_{ij}\,d_k - C_{ik}\,d_i$,

(iv) $\epsilon_{ijk}\,a_k\,b_j$,

(v) $\epsilon_{ijk}\,a_j\,a_k$.

Write the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ in suffix notation and derive an equivalent expression that utilises scalar products. Express the result both in suffix notation and in standard vector notation. Hence or otherwise determine $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and $\mathbf{c} = \mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}$.

1/I/2B **Algebra and Geometry**

Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector. Consider the operation

$$\mathbf{x} \mapsto \mathbf{n}\times\mathbf{x}\,.$$

Write this in matrix form, i.e., find a 3×3 matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{x} = \mathbf{n}\times\mathbf{x}$ for all $\mathbf{x}$, and compute the eigenvalues of $\mathbf{A}$. In the case when $\mathbf{n} = (0,0,1)$, compute $\mathbf{A}^2$ and its eigenvalues and eigenvectors.

1/II/5C **Algebra and Geometry**

Give the real and imaginary parts of each of the following functions of $z = x + iy$, with x, y real,

(i) e^z,

(ii) $\cos z$,

(iii) $\log z$,

(iv) $\dfrac{1}{z} + \dfrac{1}{\bar{z}}$,

(v) $z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3 - \bar{z}$,

where $\bar{z}$ is the complex conjugate of z.

An ant lives in the complex region R given by $|z-1| \leq 1$. Food is found at z such that

$$(\log z)^2 = -\frac{\pi^2}{16}.$$

Drink is found at z such that

$$\frac{z + \frac{1}{2}\bar{z}}{\left(z - \frac{1}{2}\bar{z}\right)^2} = 3, \quad z \neq 0.$$

Identify the places within R where the ant will find the food or drink.

1/II/6B **Algebra and Geometry**

Let $\mathbf{A}$ be a real 3×3 matrix. Define the rank of $\mathbf{A}$. Describe the space of solutions of the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,, \qquad (\dagger)$$

organizing your discussion with reference to the rank of $\mathbf{A}$.

Write down the equation of the tangent plane at $(0,1,1)$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 2$ and the equation of a general line in $\mathbb{R}^3$ passing through the origin $(0,0,0)$.

Express the problem of finding points on the intersection of the tangent plane and the line in the form $(\dagger)$. Find, and give geometrical interpretations of, the solutions.

1/II/7A Algebra and Geometry

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Show that $\mathbf{a}$ may be written as the sum of two vectors: one parallel (or anti-parallel) to $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$. By setting the former equal to $\cos\theta|\mathbf{a}|\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is a unit vector along $\mathbf{b}$, show that

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\,.$$

Explain why this is a sensible definition of the angle θ between $\mathbf{a}$ and $\mathbf{b}$.

Consider the 2^n vertices of a cube of side 2 in $\mathbb{R}^n$, centered on the origin. Each vertex is joined by a straight line through the origin to another vertex: the lines are the 2^{n-1} diagonals of the cube. Show that no two diagonals can be perpendicular if n is odd.

For $n = 4$, what is the greatest number of mutually perpendicular diagonals? List all the possible angles between the diagonals.

1/II/8A Algebra and Geometry

Given a non-zero vector v_i, any 3×3 symmetric matrix T_{ij} can be expressed as

$$T_{ij} = A\delta_{ij} + Bv_iv_j + (C_iv_j + C_jv_i) + D_{ij}$$

for some numbers A and B, some vector C_i and a symmetric matrix D_{ij}, where

$$C_iv_i = 0, \qquad D_{ii} = 0, \qquad D_{ij}v_j = 0\,,$$

and the summation convention is implicit.

Show that the above statement is true by finding A, B, C_i and D_{ij} explicitly in terms of T_{ij} and v_j, or otherwise. Explain why A, B, C_i and D_{ij} together provide a space of the correct dimension to parameterise an arbitrary symmetric 3×3 matrix T_{ij}.

3/I/1D Algebra and Geometry

Let A be a real 3×3 symmetric matrix with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Consider the surface S in $\mathbb{R}^3$ given by

$$x^TAx = 1.$$

Find the minimum distance between the origin and S. How many points on S realize this minimum distance? Justify your answer.

3/I/2D Algebra and Geometry

Define what it means for a group to be cyclic. If p is a prime number, show that a finite group G of order p must be cyclic. Find all homomorphisms $\varphi : C_{11} \to C_{14}$, where C_n denotes the cyclic group of order n. [*You may use Lagrange's theorem.*]

3/II/5D Algebra and Geometry

Define the notion of an action of a group G on a set X. Assuming that G is finite, state and prove the Orbit-Stabilizer Theorem.

Let G be a finite group and X the set of its subgroups. Show that $g(K) = gKg^{-1}$ ($g \in G$, $K \in X$) defines an action of G on X. If H is a subgroup of G, show that the orbit of H has at most $|G|/|H|$ elements.

Suppose H is a subgroup of G and $H \neq G$. Show that there is an element of G which does not belong to any subgroup of the form gHg^{-1} for $g \in G$.

3/II/6D Algebra and Geometry

Let $\mathcal{M}$ be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $SL(2, \mathbb{C})$ be the group of all 2×2 complex matrices with determinant 1.

Show that the map $\theta : SL(2, \mathbb{C}) \to \mathcal{M}$ given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}$$

is a surjective homomorphism. Find its kernel.

Show that every $T \in \mathcal{M}$ not equal to the identity is conjugate to a Möbius map S where either $Sz = \mu z$ with $\mu \neq 0, 1$, or $Sz = z \pm 1$. [*You may use results about matrices in $SL(2, \mathbb{C})$, provided they are clearly stated.*]

Show that if $T \in \mathcal{M}$, then T is the identity, or T has one, or two, fixed points. Also show that if $T \in \mathcal{M}$ has only one fixed point z_0 then $T^n z \to z_0$ as $n \to \infty$ for any $z \in \mathbb{C} \cup \{\infty\}$.

3/II/7D **Algebra and Geometry**

Let G be a group and let $Z(G) = \{h \in G : gh = hg \text{ for all } g \in G\}$. Show that $Z(G)$ is a normal subgroup of G.

Let H be the set of all 3×3 real matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that H is a subgroup of the group of invertible real matrices under multiplication.

Find $Z(H)$ and show that $H/Z(H)$ is isomorphic to $\mathbb{R}^2$ with vector addition.

3/II/8D **Algebra and Geometry**

Let A be a 3×3 real matrix such that $\det(A) = -1$, $A \neq -I$, and $A^T A = I$, where A^T is the transpose of A and I is the identity.

Show that the set E of vectors x for which $Ax = -x$ forms a 1-dimensional subspace.

Consider the plane Π through the origin which is orthogonal to E. Show that A maps Π to itself and induces a rotation of Π by angle θ, where $\cos\theta = \frac{1}{2}(\text{trace}\,(A) + 1)$. Show that A is a reflection in Π if and only if A has trace 1. [*You may use the fact that* $\text{trace}(BAB^{-1}) = \text{trace}(A)$ *for any invertible matrix* B.]

Prove that $\det(A - I) = 4(\cos\theta - 1)$.